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Designating the right member of this inequality by u_{r+1} , we see that Σu_r is convergent, since

$$\lim_{r=\infty} \frac{u_{r+1}}{u_r} = \lim_{r=\infty} \frac{9}{10} \left(\frac{r+1}{r} \right)^{p-1} = \frac{9}{10} < 1.$$

Hence Σg_r is convergent, and therefore also S is convergent.

NOTE.—One may generalize the proposition thus:

The series S' whose terms are the reciprocals of all positive integers not containing any combination C whatever of the digits 0, 1, 2, \dots , 9 (which contains at least one digit) is convergent.

For, suppose C contains p digits. It may be shown that there are not more terms in a group g'_{r+1} in this series than in the corresponding group g_{r+1} previously discussed. When this is established the preceding argument proves the proposition.

When the terms of the series are the reciprocals of all the positive integers the series diverges. The proposition states that if from this divergent series certain terms are stricken out a convergent series is obtained. This is not particularly surprising when one observes that obviously a very large percentage of the integers of a thousand places, for example, contain a zero, and hence most of the terms far out in the series are stricken out.

Note.—This solution was received before the appearance of the article by Dr. Irwin on "A curious convergent series" in the May MONTHLY. Dr. Irwin there proves this generalized theorem by a somewhat different style of argument. He gives on page 50 some details of the reasoning by which equation (1) above may be established.—EDITORS.

Also solved by A. H. HOLMES and G. W. HARTWELL.

454. Proposed by C. N. SCHMALL, New York City.

Prove that a number is divisible by nine if, and only if, the sum of its digits is divisible by nine.

SOLUTION BY FRANK R. MORRIS, Glendale, California.

Let a, b, c, d, \dots be the digits of any whole number, in order from right to left. Then this number may be written in the form

$$a + 10b + 100c + 1,000d + \dots$$

Dividing by 9 the quotient is

$$a/9 + (1 + 1/9)b + (11 + 1/9)c + (111 + 1/9)d + \dots,$$

which equals

$$(b + 11c + 111d \dots) + \frac{a + b + c + d + \dots}{9}.$$

The first term of this quotient is integral and the second term is integral only if $a + b + c + d + \dots$ is divisible by 9. Therefore, the number is divisible by nine if, and only if, the sum of its digits is divisible by nine.

This proof holds for a number containing a decimal fraction, as may be seen by converting the number into a whole number divided by some power of ten.

The above number may also be written

$$a + (9 + 1)b + (99 + 1)c + (999 + 1)d + \dots$$

or

$$9b + 99c + 999d + \dots + (a + b + c + d + \dots),$$

which is a multiple of 9 when $(a + b + c + d + \dots)$ is a multiple of 9.

Also solved by PAUL CAPRON, ELIJAH SWIFT, J. W. CLAWSON, G. L. WAGAR, H. C. FEEMSTER, W. F. RIGGS, NATHAN ALTSHILLER, E. F. CANADAY, C. A. BARNHART, O. S. ADAMS, G. W. HARTWELL, A. W. SMITH, E. E. WHITFORD, HORACE OLSON, A. H. HOLMES, W. J. THOME, H. S. UHLER, and H. N. CARLETON.

GEOMETRY.

475. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

SOLUTION (COMPLETED) BY HORACE OLSON, Chicago, Illinois.

Professor Hartwell's solution of this problem in the June issue of the MONTHLY is incomplete; it does not cover the case in which the given line is parallel to the radical axis of the circles.

This case may be considered in two parts, according as the given circles do or do not intersect in real points. If they intersect in real points, the required circle passes through three known real points, and can therefore readily be constructed.

If the given circles do not intersect in real points, we have $x^2 - r^2 = c^2$, where x is the distance from the radical axis to the center of any circle of the system of coaxial circles, r is its radius, and c is a real constant ascertainable from the given circles; c is, in fact, the length of the tangent from the intersection of the line of centers and the radical axis to any circle of the system. The distance d from the radical axis to the given line is either $x + r$ or $x - r$, according as it is $\geq c$, x and r being here the x and r of the required circle. From these two equations, we find

$$x = \frac{d}{2} + \frac{c^2}{2d}, \quad \text{and} \quad r = \left| \frac{d}{2} - \frac{c^2}{2d} \right|.$$

Thus, the required circle is unique and can easily be constructed. Professor Hartwell's second solution becomes, in the cases I am considering, $x = \infty$, $r = \infty$; i. e., the radical axis itself.

481. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that the locus of the intersection of a pair of perpendicular normals to a parabola $y^2 = 4px$ is the parabola $y^2 = p(x - 3p)$.

SOLUTION BY F. M. MORGAN, Dartmouth College.

The line $y = mx - 2pm - pm^3$ is a normal to the parabola for all values of m . If x and y are considered as constants, then the roots of this equation are the directions of the normals that pass through the point (x, y) . Call the roots m_1, m_2, m_3 . Now, $m_1 m_2 m_3 = -(y/p)$. But if two normals are perpendicular, $m_1 m_2 = -1$. Therefore, $m_3 = (y/p)$.

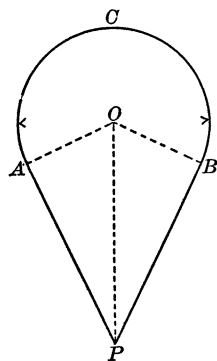
However, $y = m_3 x - 2pm_3 - pm_3^3$; i. e.,

$$y = \frac{xy}{p} - 2y - \frac{y^3}{p^2},$$

which when simplified takes the form $y^2 = p(x - 3p)$.

Also solved by NORMAN ANNING, H. R. HOWARD, ELIJAH SWIFT, C. K. ROBBINS, J. A. BULLARD, H. L. AGARD, S. E. RASOR, T. O. WALTON, R. W. LORD, HORACE OLSON, C. M. SPARROW, G. W. HARTWELL, R. A. JOHNSON, C. A. EPPERSON, C. E. DIMICK, H. S. UHLER, and the PROPOSER.

482. Proposed by ROBERT G. THOMAS, The Citadel, Charleston, S. C.



In laying out a kite-shaped mile race-track, composed of a circular arc and two intersecting tangents at the ends of the arc, determine the angle at the center of the arc (a) when the length of the arc equals the sum of the two tangents, and (b) when the arc is equal to the length of each tangent.

SOLUTION BY WILLIAM W. JOHNSON, Cleveland, Ohio.

Let AP and BP be the two tangents, and ACB the arc composing the race-track; O the center, and OB the radius of the arc. Let $BOP = \varphi$, $OB = r$, and $AP = BP = t$. Then, the length of the arc $ACB = r(2\pi - 2\varphi)$. By the conditions in (a), (1) $2r(\pi - \varphi) = 2t$, and (2) $r = t/\tan \varphi$. Eliminating r between (1) and (2), we obtain (3) $\varphi + \tan \varphi = \pi$. By the conditions in (b), we have (4) $2r(\pi - \varphi) = t$. Eliminating r between (2) and (4), we obtain (5) $2\varphi + \tan \varphi = 2\pi$. Solving equations (3) and (5) by the Method of Successive Approxima-

tions, we find from (3), angle $\varphi = 63^\circ 45' 38.657''$. Whence, angle at center of arc $= 2(\pi - \varphi) = 232^\circ 28' 42.686''$, answer to (a). From (5), angle $\varphi = 74^\circ 46' 14.636''$. Whence, angle at center of arc $= 2(\pi - \varphi) = 210^\circ 27' 30.728''$, answer to (b).